J. Fluid Mech. (1990), *vol.* 212, *pp.* 637–662
Printed in Great Britain

Mean fields and fluctuation moments in unstably stratified turbulent boundary layers

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(Received 15 June 1989)

The earliest results concerning the turbulence structure in a turbulent boundary layer with very unstable thermal stratification are due to Prandtl (1932). These results were developed further and made more precise by Obukhov (1946, 1960), Monin & Obukhov (1954) and Priestley (1954, 1955, 1956, 1960). All of these authors dealt with a surface layer of the Earth's atmosphere on hot summer days. Such a layer is the most easily accessible example of an unstably stratified boundary layer and it will be the main concern in this paper too. The theoretical predictions by the above-mentioned authors seemed at first to be confirmed by the available experimental data but in the late 1960s it became clear that at least some of the predictions disagreed strongly with the experimental information.

A more elaborate theory was proposed by Betchov & Yaglom (1971) who used a suggestion of Zilitinkevich (1971). According to this theory, within an unstably stratified boundary layer there are three special sublayers where turbulence structure is self-preserving and obeys rather simple power laws. The new theory explained the disagreement between some of the deductions from the old theory and the data. However, the data available in 1971 were insufficient for the confirmation of the new theory and it was even supposed by Betchov & Yaglom (1971) that their theory could *not* be applied to atmospheric surface layers on hot summer days.

Much new experimental data concerning unstably stratified boundary layers has been obtained in recent years; in particular, extensive experimental information was collected during the summers of 1981–1987 at the Tsimlyansk Field Station of the Moscow Institute of Atmospheric Physics. This paper is a survey of the deductions from the theory by Betchov & Yaglom which concern the mean fields and the one-point fluctuation moments in unstably stratified boundary layers, and a comparison of these deductions with the data available in 1989. It is shown that the data agree more or less satisfactorily with the theoretical predictions and permit one to obtain estimates for a number of coefficients that enter the theoretical equations.

1. Introduction

It is known that the nonlinearity of the equations of fluid mechanics makes the dynamic equations for both the mean fields and any fluctuation moments of turbulent flows non-closed, i.e. the number of unknowns in the equations always exceeds the number of available equations. Therefore these equations cannot be solved and in practice they are usually replaced by some model equations obtained by supplementing the rigorous dynamic equations by some speculative closure hypotheses. Such model equations are, of course, not strict and the estimation of their accuracy and the selection of the best closure hypotheses is a very difficult task.

Therefore, results pertaining to turbulent flow that can be obtained without any use of dynamic equations and closure hypotheses are of a special interest.

The most important general method for obtaining meaningful physical results without solving the dynamic equations is based on similarity and dimensional arguments. These arguments have often been fruitfully applied to the mechanics of turbulence; typical examples are Kolmogorov and Obukhov's theory of locally isotropic turbulence (see e.g. Batchelor 1953 or Monin & Yaglom 1975, Chap. 8), similarity theory of turbulence in thermally stratified boundary layers due to Monin & Obukhov (1954), and similarity laws of turbulent diffusion studied by Batchelor (1950, 1952, 1964). This paper presents some new developments of the Monin & Obukhov (1954) theory based on refined dimensional considerations.

We shall consider two-dimensional turbulent boundary layers in fluid of variable density flowing over a horizontal flat wall at $z = 0$ in the Ox-direction and having an unstable density stratification, i.e. the mean density $\rho = \rho(z)$ increasing with z. The atmospheric surface layer above flat homogeneous ground on a summer day, when the ground is heated by the sun and the air temperature is decreasing with height, is such a boundary layer. An unstable boundary layer can also be produced in the laboratory by fluid of lower temperature flowing over a heated plate. In both cases the density profile $\rho(z)$ is specified by the mean temperature profile $T(z)$; therefore it will be convenient to consider below the mean fluid temperature $T = T(z)$ instead of the mean density $\rho(z)$.

Unstable turbulent boundary layers have been studied mostly by meteorologists dealing with atmospheric surface layers. Apparently the first theoretical study of such a layer is due to Prandtl (1932). He used a semiempirical mixing-length theory and came to the conclusion that

$$\frac{\mathrm{d}T}{\mathrm{d}z} \propto z^{-\frac{4}{3}}, \quad K_T \propto z^{\frac{4}{3}}, \quad W_* \propto z^{\frac{1}{3}} \tag{1.1}$$

in very unstable stratification, where $K_T = K_T(z)$ is the eddy temperature diffusivity and W_* is the typical scale of the vertical velocity fluctuations. To determine the wind velocity profile $U(z)$, Prandtl assumed that eddy viscosity K_U is proportional to K_T (i.e. $K_T = \alpha K_U$, where α is a constant). Then by virtue of the second of equations (1.1)

$$K_U(z) \propto z^{\frac{4}{3}}, \quad \frac{\mathrm{d}U}{\mathrm{d}z} \propto z^{-\frac{4}{3}}. \tag{1.2}$$

Prandtl's paper was much ahead of its time and in 1932 it did not attract much attention. The next important study of a thermally stratified atmospheric surface layer is due to Obukhov (1946). (This paper was finished in 1943, but its publication was delayed by World War II; see Businger & Yaglom 1971.) Obukhov also used the mixing-length theory but purely dimensional arguments played an important part in his reasoning. He studied the atmospheric surface layer (ASL), i.e. the lowest air layer where the Earth's rotation and the total thickness z_i of the planetary boundary layer hardly affect the flow. Within the ASL the turbulent fluxes of momentum and temperature $\tau = \rho\langle -uw\rangle$ and $Q = \langle wt\rangle$ (where t, u and w are the fluctuations of temperature and velocity components in the Ox- and Oz-directions and angular brackets symbolize averaging) can be considered as independent of z. (The thickness z_s of the ASL is usually determined in the middle latitudes from two conditions: $z \lesssim 50$ m and $z \lesssim 0.1 z_i$ within the ASL.) Obukhov clearly understood that the turbulence structure of the ASL is fully determined by the values of the momentum flux τ (or, equivalently, the friction velocity $u_* = \langle -uw \rangle^{\frac{1}{2}}$), the temperature flux Q and a buoyancy parameter $\beta = g\gamma$, where g is the acceleration due to gravity and γ is the coefficient of thermal expansion (which is equal to $1/T_0$ for an ideal gas, T_0 being the mean absolute temperature of the surface layer). In this relation he introduced the important lengthscale $L = u_*^3/Q\beta$ and showed that buoyancy forces are unimportant for $z \ll L$. Moreover, he also assumed that $K_T(z) = \alpha K_U(z)$, α = constant, and deduced from this assumption that $K_U(z) \propto K_T(z) \propto z^{\frac{4}{3}}$ for $z \gg L$ (i.e. for strong instability). The latter relations clearly imply equations (1.1) and (1.2) for $\mathrm{d}T/\mathrm{d}z$ and $\mathrm{d}U/\mathrm{d}z$. Note that Obukhov included the numerical factor $-1/\kappa$, where $\kappa \approx 0.4$ is the von Kármán constant, in the definition of the lengthscale, i.e. he used $L_O = -u_*^3/\kappa Q\beta$ instead of L. The length L_O is usually called the Obukhov or the Monin–Obukhov length and it is used in many papers and books dealing with boundary-layer meteorology. However, the length L, which is positive in unstable stratification and does not include the empirical value of κ, seems to be more convenient and it is used instead of L_O in this paper.

Some vertical profiles of dimensionless turbulence parameters in the ASL were represented by Obukhov (1946) as functions of the dimensionless height $\zeta_O = z/L_O$ (easily replaceable by $\zeta = z/L = -\zeta_O/\kappa$). He derived such representations from some model (semiempirical) equations of turbulence mechanics. However, later, Monin & Obukhov (1954) used neither dynamic equations nor semiempirical hypotheses but based all their conclusions only on general similarity and dimensional arguments. They considered the flow region not too close to the ground where the molecular transfers and ground roughness do not affect directly the turbulence structure and gave the following general equations for the wind velocity and temperature gradients:

$$\frac{\mathrm{d}U}{\mathrm{d}z} = \frac{u_*}{z}\phi_U\left(\frac{z}{L}\right), \quad \frac{\mathrm{d}T}{\mathrm{d}z} = -\frac{T_*}{z}\phi_T\left(\frac{z}{L}\right), \tag{1.3}$$

where $T_* = Q/u_*$, and $\phi_U(\zeta)$, $\phi_T(\zeta)$ are universal functions of $\zeta = z/L$. If $z \ll L$, then we can assume that $L = u_*^3/Q\beta = \infty$, i.e. $\beta = 0$. Therefore $\mathrm{d}U/\mathrm{d}z$ and $\mathrm{d}T/\mathrm{d}z$ must be independent of β for $z \ll L$. Thus,

$$\phi_U(\zeta) \approx \phi_U(0) = A_U, \quad \phi_T(\zeta) \approx \phi_T(0) = A_T \quad \text{for } \zeta \ll 1, \tag{1.4}$$

where A_U and A_T are constants. (It is easy to see that $A_U = 1/\kappa$, $A_T = A_U P_t$, where $\kappa \approx 0.4$ is the von Kármán constant and $P_t \approx 0.85$ is the turbulent Prandtl number within the logarithmic sublayer of a turbulent boundary layer; see e.g. Yaglom 1979, or Kader & Yaglom 1980.) Moreover, if $z \gg L$, then $\mathrm{d}T/\mathrm{d}z$ and $K_T = -Q/[\mathrm{d}T/\mathrm{d}z]$ must be independent of u_* (since passage to the limit $L \to 0$ is equivalent to $u_* \to 0$). Therefore $\phi_T(\zeta) = -B_T\zeta^{-\frac{1}{3}}$, where B_T = const., for $\zeta \gg 1$ and, in accordance with (1.1),

$$\frac{\mathrm{d}T}{\mathrm{d}z} = -B_T Q^{\frac{2}{3}}\beta^{-\frac{1}{3}}z^{-\frac{4}{3}}, \quad K_T = B_T^{-1}(Q\beta)^{\frac{1}{3}}z^{\frac{4}{3}} \quad \text{for } z \gg L. \tag{1.5}$$

In addition, Monin & Obukhov suggested (as did Prandtl 1932 and Obukhov 1946) that $K_T(z) = \alpha K_U(z)$, $K_U\,\mathrm{d}U/\mathrm{d}z = u_*^2$, where α = const. and u_* = const.; this suggestion implies that

$$\frac{\mathrm{d}U}{\mathrm{d}z} = B_U u_*^2(Q\beta)^{-\frac{1}{3}}z^{\frac{4}{3}}, \quad B_U = \text{const.} \quad \text{for } z \gg L \tag{1.6}$$

(i.e. that (1.2) is valid).

For one-point moments of velocity component and temperature fluctuations u, v, w and t, the dimensional analysis gives equations similar to (1.3). In particular,

$$\left.\begin{aligned}\sigma_u = \langle u^2\rangle^{\frac{1}{2}} = u_* \phi_1(z/L),\ \sigma_v = \langle v^2\rangle^{\frac{1}{2}} = u_* \phi_2(z/L),\ \sigma_w = \langle w^2\rangle^{\frac{1}{2}} = u_* \phi_3(z/L),\\ \sigma_t = \langle t^2\rangle^{\frac{1}{2}} = T_* \phi_4(z/L),\ \langle ut\rangle = -Q\phi_5(z/L),\end{aligned}\right\} \quad (1.7)$$

where ϕ_i, $i = 1, \dots, 5$, are universal functions of $\zeta = z/L$. The moments (1.7) must be independent of β for $z \ll L$; therefore all functions $\phi_i(\zeta)$ must be constant for $\zeta \ll 1$. On the other hand it is natural to assume that the moments (1.7) must be independent of u_* for $z \gg L$. This assumption implies that

$$\phi_1 \sim \zeta^{\frac{1}{3}}, \quad \phi_2 \sim \zeta^{\frac{1}{3}}, \quad \phi_3 \sim \zeta^{\frac{1}{3}}, \quad \phi_4 \sim \zeta^{-\frac{1}{3}}, \quad \phi_5 = \text{const.} \quad (1.8)$$

for $\zeta \gg 1$ so that

$$\left.\begin{aligned}\sigma_u = C_1(Q\beta z)^{\frac{1}{3}}, \quad \sigma_v = C_2(Q\beta z)^{\frac{1}{3}}, \quad \sigma_w = C_3(Q\beta z)^{\frac{1}{3}},\\ \sigma_t = C_4 Q^{\frac{2}{3}}(\beta z)^{-\frac{1}{3}}, \quad \langle ut\rangle = -C_5 Q\end{aligned}\right\} \quad (1.9)$$

at $z \gg L$, where $C_1, \dots, C_5$ are universal constants.

Equations (1.9) for σ_u, σ_v, σ_w and σ_t were given by Obukhov (1960) who assumed in addition that $C_1 = C_2$; the equation for σ_w is clearly related to Prandtl's equation (1.1) for W_*. Equation (1.9) for σ_t was also obtained by Priestley (1960) and in fact it follows from the results of Priestley (1954, 1955, 1956) which were mainly devoted to a discussion and comparison with the data of equation (1.5) for dT/dz.

The early Australian and USSR measurements of wind velocity and temperature profiles in an unstable ASL convey the impression that (1.5) and (1.6) agree well with the data for a height range beginning at an unexpectedly small value of z, on the order of $0.1L$; see e.g. Priestley (1959) and Monin & Yaglom (1971, §8). (The subsequent data show that in fact (1.6) begins to be valid at slightly larger values of z than (1.5) but this is of minor importance.) The measurement of the second-order moments of turbulent fluctuations is more complicated than that of the mean fields; nevertheless, measurements of σ_t and σ_w were quite often taken in the ASL during the 1960s. The results obtained show that the equations (1.9) for σ_t and σ_w agree well with the data over a wide range of z-values larger than $0.1L$. Note also that measurement of σ_u and σ_v in the ASL presents great difficulties and till now the results have been rather unreliable, while measurements of the moment $\langle ut\rangle$ first appeared only in the late 1960s. Therefore in the middle of the 1960s it was believed that the simple results (1.5), (1.6) and (1.9) were confirmed by the data for $z \gtrsim 0.1L$.

In fact, however, the situation was not so clear then. First of all it was a little strange that the asymptotic equations for $\zeta \gg 1$ proved to be valid at very small values of ζ, of the order of 0.1. (Note that in all the literature the variable $\zeta_0 = z/L_0 = -\kappa\zeta$ was used instead of ζ; therefore it was believed that the asymptotic equations for $\zeta_0 \ll -1$ proved to be valid at $\zeta_0 \approx -0.04$.) Moreover, the derivation of (1.5), (1.6) and (1.9) could not be considered as fully satisfactory, since the derivation of (1.5) and (1.9) was based on the assumption that u_* does not affect the turbulence structure at $z \gg L$, even though (1.6) includes u_*. Also, it became clear in the second half of the 1960s that at greater values of ζ, on the order of a few units, the agreement of the results (1.5) and (1.6) with the experimental data becomes rather poor (see e.g. Monin & Yaglom 1971, p. 499, and Businger *et al.* 1971). Finally, the first measurements of the vertical profiles of the moments $\langle ut\rangle$ and $\langle uwt\rangle$ in the unstable surface layer performed in the late 1960s gave results (described by Monin

& Yaglom 1971, pp. 522–523, and Wyngaard, Coté & Izumi 1971) that strongly disagreed with the last of equations (1.9) and the related result for $\langle uwt \rangle$ implied by the assumption that this moment is independent of u_* for $z \gtrsim L$.

The inconsistency of the derivation of (1.5) and (1.6) by Prandtl, Obukhov and Monin lends interest to Bernstein's (1966) attempt to apply the so-called 'directional dimensional analysis' to the study of the mean velocity, temperature and humidity profiles in the ASL. (This generalized form of dimensional analysis will be discussed in §2 of this paper.) In fact, Calder (1967) showed that directional analysis automatically leads to equations (1.5) and (1.6). However, Calder also showed that this analysis does not give correct results for neutral stratification and therefore he concluded that the analysis is incorrect. (Bernstein agreed with this conclusion in his reply to Calder's comments.) Nevertheless later Zilitinkevich (1971) tried again to apply directional dimensional analysis to the study of an unstable ASL. He showed that the analysis not only implies (1.5) and (1.6) but also leads to some other plausible results, e.g. to an equation for $\langle ut \rangle$ which, for $\zeta \gtrsim 0.1$, agrees satisfactorily with the available data. In their comments on Zilitinkevich's paper Betchov & Yaglom (1971) reanalysed the Zilitinkevich assumptions and developed a three-sublayer model of unstably stratified turbulent boundary layers. According to Betchov & Yaglom's arguments, (1.5), (1.6) and the related equations for $\langle ut \rangle$ and $\langle uwt \rangle$ implied by directional analysis are valid only in a restricted range of moderate values of ζ, while for very large values of ζ different equations hold true. This prediction agrees qualitatively with the observed deviations of the wind velocity and temperature profiles from the laws (1.5) and (1.6) at $\zeta \approx 2$. However, in 1971 micrometeorological data were available only for $\zeta \lesssim 5$ and therefore they were clearly insufficient for reliable verification of the predictions by Betchov & Yaglom, and there was doubt about the applicability of their predictions to the ASL.

Many additional measurements of turbulence structure for unstable stratified boundary layers were carried out in subsequent years both in the atmosphere and laboratory flows. In particular, extensive data for a wide range of positive values of ζ were collected from the atmospheric experiments performed at the Tsimlyansk Field Station of the Moscow Institute of Atmospheric Physics during the summers of 1981–1987. The new data agree quite satisfactorily with the three-sublayer model of 1971 and permit one to determine the ranges for the three sublayers and to estimate many empirical coefficients entering the theoretical equations. This paper is a survey of theoretical results and experimental data concerning the mean fields and fluctuation moments in unstably stratified ASL and laboratory turbulent boundary layers (parts of which are briefly described by Kader & Yaglom 1984, Kader 1988 and Kader & Perepelkin 1989). Note that only convective boundary layers with shear will be considered in the paper; the case of shear-free convective boundary layers studied, for example, by Hunt (1984) is rather different and will not be touched on below. (However, the comparison of the atmospheric data with the vertical profiles of horizontally averaged quantities in fully turbulent laboratory convection seems to be rather interesting.)

The three-sublayer model can also be applied to multipoint statistical parameters, for example to spectra and correlation functions of turbulent fluctuations. Some deductions from the model concerning the longitudinal (along the mean wind direction) spatial spectra and correlation functions of turbulence within the unstable ASL were given by Kader (1987, 1988), Kader & Yaglom (1984, 1987) and Kader, Yaglom & Zubkovskii (1989), but this material is beyond the scope of this paper.

2. Directional dimensional analysis and the three-sublayer model for an unstable boundary layer

Theoretical study of the ASL profiles of meteorological quantities by Bernstein (1966), Zilitinkevich (1971, 1973) and Betchov & Yaglom (1971) is based on the application of the so-called 'directional' (or 'anisotropic', or 'vector') dimensional analysis to the problem under consideration. This form of dimensional analysis was first developed in the nineteenth century and, although it is often applied to specific physical problems, it continues to be controversial and to raise many doubts. The analysis uses different dimensions for lengths in different (orthogonal) directions, for example dimensions L_x and L_z for horizontal and vertical lengths or three dimensions L_x, L_y, L_z for lengths in three mutually perpendicular directions. The increase in the number of basic dimensions clearly decreases the number of dimensionless combinations of relevant physical parameters; therefore, it permits one to obtain sharper results than those implied by the conventional dimensional analysis. This makes directional dimensional analysis very attractive but the conditions that guarantee the correctness of its deductions are far from obvious.

Williams (1892) was apparently the first who proposed to apply dimensional analysis with several length dimensions to physical problems. Later such an analysis was used (or, at least, mentioned) in many papers and a number of books (e.g. by Huntley 1952; Isaacson & Isaacson 1975; Barenblatt 1980; Panton 1984 and Kline 1986). Williams, Huntley and Kline did not discuss at all the conditions for the applicability of such an analysis but gave only some specific examples where it led to correct results. It is, however, easy to find other examples where its consequences are clearly incorrect. (This is precisely the reason why Calder 1967 rejected Bernstein's arguments of 1966.) Barenblatt (1980, §7.2) presented an example where the applicability of directional analysis follows from the invariance of dynamic equations with respect to some subgroup of affine transformations, but he did not discuss the problem in detail. Panton (1984, §8.8) noted that the use of several length dimensions works when the physical processes for different directions are independent of each other; however, in his opinion such extra information can also always be used in a more direct manner. Isaacson & Isaacson (1975) assert that the use of different length dimensions for different directions must be based on 'orthogonal independence' of relevant physical processes but they do not give a strict definition of the term used. A more detailed discussion of the problem was given by Betchov & Yaglom (1971) and Massey (1978) whose conclusions coincide only partially.

In both the above-mentioned papers it is noted that the increase of the number of 'basic dimensions' by one would not change the conclusions of the dimensional analysis, provided one more special physical parameter is added to the list of parameters relevant to the problem. The new parameter must be chosen so that its dimension coincides with the ratio of the new 'basic dimension' to the 'conventional dimension' of the corresponding physical parameter. Specifically, Betchov & Yaglom (1971) considered the case where the sole length dimension L_1 is replaced by two dimensions L_x and L_z of the horizontal and vertical lengths, and stated that then no incorrect conclusions will be obtained, provided a new parameter of dimension L_z/L_x (or, equivalently, $(L_z/L_x)^n$) is added to the list of relevant physical parameters. Similarly, Massey (1978) explained that the transition from the three basic dimensions M_1, L_1 and T_1 of mass, length and time to four basic dimensions M_1, L_1, T_1 and F_1, where F_1 is the dimension of force, would not change all the dimensional equations, provided a new relevant physical parameter of dimension $F_1/M_1 L_1 T_1^{-2}$ is additionally taken into account. However, Betchov & Yaglom considered the parameter of dimension $(L_z/L_x)^n$ as a physical quantity characterizing the rate of energy exchange between horizontal and vertical motions, while Massey introduced the parameter of dimension $F_1/M_1 L_1 T_1^{-2}$ purely formally. Betchov & Yaglom's approach implies the conclusion that the 'additional relevant parameter' vanishes (i.e. it must not be taken into account), if horizontal and vertical motions are energetically uncoupled (i.e. there is no energy exchange between them). Therefore in this case directional analysis can be used without modifying the list of relevant parameters (this conclusion was also stated by Zilitinkevich 1971). On the other hand Massey suggested that two independent length dimensions L_x and L_z can be used only in cases where there is no force acting in either the x- or z-direction (i.e. there is no energy influx or outflux in one of the directions); therefore he concluded that directional dimensional analysis cannot be applied to a turbulent flow (the same statement was formulated by Calder 1967). Massey's condition is, of course, a particular case of that of Betchov & Yaglom. It seems, however, to be clear that just the energy exchange among processes in different directions and not the presence of forces acting in these directions makes directional analysis incorrect without extension of the list of relevant parameters. Therefore we shall base our consideration in this paper on Betchov & Yaglom's condition. It will be demonstrated below that this condition, when applied to turbulence in an unstable boundary layer, implies quite plausible results which do not contradict the available data.

Let us consider first a turbulent boundary layer in neutral (or almost neutral) thermal stratification. Assume that the distance z from the wall is large in comparison with the molecular transfer and roughness lengths ν/u_*, D/u_* and h_0 (where ν and D are molecular viscosity and temperature diffusivity and h_0 is the mean height of the wall roughness elements) but is small in comparison with the boundary-layer or ASL thickness. (Below, this condition will be always assumed to be true.) Then, as is well known,

$$\frac{\mathrm{d}U}{\mathrm{d}z} = \frac{u_*}{\kappa z}, \quad \frac{\mathrm{d}T}{\mathrm{d}z} = -\frac{T_* P_\mathrm{t}}{\kappa z}, \tag{2.1}$$

where $\kappa \approx 0.4$, $P_\mathrm{t} \approx 0.85$ (see (1.3) and (1.4)). Let us apply directional dimensional analysis with different dimensions L_x and L_z of horizontal and vertical lengths to equations (2.1). Then obviously $[U] = L_x T_1^{-1}$, $[u_*] = L_x^{\frac{1}{2}} L_z^{\frac{1}{2}} T_1^{-1}$ (where the square brackets denote dimension and T_1 is the time dimension). Hence directional analysis implies that κ is not a dimensionless constant but is a physical quantity of dimension $L_z^{\frac{1}{2}} L_x^{-\frac{1}{2}}$. According to the experimental data $\kappa \approx 0.4$ if (and only if) the horizontal and vertical lengths are measured by the same units.

Consider now the budgets of the kinetic energy components in high-Reynolds-number turbulence. For a steady two-dimensional stratified boundary layer they have the form

$$\left.\begin{aligned}
\langle -uw\rangle \frac{\mathrm{d}U}{\mathrm{d}z} + \rho^{-1}\left\langle p\frac{\partial u}{\partial x}\right\rangle &= \epsilon_u + \frac{1}{2}\frac{\partial\langle u^2 w\rangle}{\partial z},\\
\rho^{-1}\left\langle p\frac{\partial v}{\partial y}\right\rangle &= \epsilon_v + \frac{1}{2}\frac{\partial\langle v^2 w\rangle}{\partial z},\\
Q\beta + \rho^{-1}\left\langle p\frac{\partial w}{\partial z}\right\rangle &= \epsilon_w + \frac{1}{2}\frac{\partial\langle w^3 + 2pw/\rho\rangle}{\partial z},
\end{aligned}\right\} \tag{2.2}$$

where p is the pressure fluctuation and ϵ_u, ϵ_v and ϵ_w are the rates of viscous dissipation for $\frac{1}{2}\langle u^2\rangle$, $\frac{1}{2}\langle v^2\rangle$ and $\frac{1}{2}\langle w^2\rangle$; see e.g. Monin & Yaglom (1971, §6). Here $\langle -uw\rangle \mathrm{d}U/\mathrm{d}z$ is the rate of dynamic production of the energy of longitudinal velocity fluctuations (which borrow energy from that of the mean motion) and $Q\beta = \langle wt\rangle \beta$ is the rate of thermal production of the energy of vertical fluctuations by buoyancy forces, the terms on the right-hand sides of (2.2) describe the viscous dissipation and the vertical transfer of energy components, while the terms containing p on the left-hand sides determine the rates of energy exchange among the components.

According to (2.1) $\langle -uw\rangle \mathrm{d}U/\mathrm{d}z \approx u_*^3/\kappa z$ if the thermal stratification is close to neutral. Obukhov (1946) assumed that buoyancy forces play no role if the rate of energy production by these forces is smaller than that of dynamic energy production, i.e. if $z < -L_{\mathrm{O}}$, where $L_{\mathrm{O}} = -u_*^3/\kappa\beta Q$. However, in the framework of directional dimensional analysis this conclusion should be modified. According to such an analysis the rates of energy production $u_*^3/\kappa z$ and $Q\beta$ have different dimensions, $L_x^2 T_1^{-3}$ and $L_z^2 T_1^{-3}$; therefore their ratio $-L_{\mathrm{O}}/z$ is not a dimensionless quantity but has the dimension $L_x^2 L_z^{-2}$. The buoyancy forces do not affect the turbulence structure only if the rate of the influx of energy to the vertical velocity component from the longitudinal one due to the action of the pressure fluctuations is greater than the rate of direct production of $\frac{1}{2}\langle w^2\rangle$ by buoyancy forces. Moreover, the rate of the dynamic energy influx to the vertical velocity fluctuations must be of the order of the unique combination of u_*, z and κ having the same dimension $L_z^2 T_1^{-3}$ as $Q\beta$, i.e. this rate must be of the order of $(\kappa u_*)^3/z$. These arguments make us think that the *dynamic sublayer* of an unstably stratified boundary layer (i.e. the sublayer where the buoyancy forces play no part) is determined by the condition $z < L_*$, where $L_* = (\kappa u_*)^3/Q\beta$ is a vertical lengthscale which is approximately 40 times smaller than the original Obukhov length $|L_{\mathrm{O}}| = u_*^3/\kappa\beta Q$ and 15 times smaller than $L = u_*^3/Q\beta$ (since $\kappa^{-4} \approx 40$, $\kappa^{-3} \approx 15$). This estimate of the dynamic sublayer thickness was proposed by Betchov & Yaglom (1971); it agrees rather well with the observations.

For $z > L_*$ inclusion of the thermal production becomes essential and therefore the parameter β must be taken into account. If z only slightly exceeds L_*, then the buoyancy forces affect the w-component but the u- and, probably, also v-fluctuations are produced dynamically and part of the energy of the u-fluctuations borrowed from the mean motion is transferred to the w-fluctuations by means of pressure fluctuations. Moreover, a range of z-values exists somewhere above the level $z = L_*$ where the energy influx to the w-fluctuations from the energy of the mean motion by the action of the pressure fluctuations becomes negligibly small as compared to the thermal production of the vertical energy, but the horizontal velocity fluctuations have, nevertheless, a purely dynamic origin. Within this range the values of z/L_* must be greater than one, but not too large, and for such z-values the rate of the interconversion of the energies of the vertical and horizontal fluctuations is very small compared with both the rates of energy production $\langle -uw\rangle \mathrm{d}U/\mathrm{d}z$ and $Q\beta$. It is natural to think that in the corresponding flow region this interconversion can be neglected to a first approximation, and the vertical and horizontal motions can be considered to be energetically uncoupled. Therefore the directional analysis (with two length dimensions L_z and L_x) can be applied here without any extension of the list of relevant parameters. The region of an unstable boundary layer considered will be called below the *dynamic–convective sublayer*.

At still greater heights the rate of convective energy production exceeds the rate of dynamic energy production and the energy transfer from the vertical to horizontal velocity fluctuations becomes significant. In the lower part of this region both the dynamic and convective production of the energy of horizontal fluctuations must be taken into account. However, the mean velocity gradient $\mathrm{d}U/\mathrm{d}z$ decreases with z and, beginning from some value of z, the dynamic energy production $\langle -uw \rangle \, \mathrm{d}U/\mathrm{d}z$ becomes so small that it can be neglected. Above this level $u_* = \langle -uw \rangle^{\frac{1}{2}}$ can be excluded from the list of relevant physical parameters. The corresponding flow region will be called the *free convection sublayer*. In this sublayer we can use either conventional dimensional analysis with one length dimension or directional analysis with two length dimensions L_z and L_x under the condition that the list of relevant physical parameters is supplemented by a parameter of dimension of the form $(L_z/L_x)^n$. The upper edge of the free convection sublayer in laboratory flows coincides with the value of z where the influence of the boundary-layer thickness becomes significant and in the atmosphere it coincides with the upper boundary of the ASL.

The three-sublayer model of an unstable turbulent boundary layer assumes that such a boundary layer has three quite distinct sublayers (dynamic, dynamic–convective and free convective) where some specific simple laws must be valid. The model was proposed by Betchov & Yaglom (1971) when there were no data to compare the model with. Note that Betchov & Yaglom (1971), Yaglom (1974) and Monin & Yaglom (1975, pp. 853–854), assumed that the free convection sublayer probably usually does not exist in the Earth's atmosphere since its lower edge is above the upper edge of the ASL; a similar point was also made by Wyngaard *et al.* (1971, p. 1172). However at present much data exists from atmospheric observations which agree quite satisfactorily with the three-sublayer model; there are also some laboratory data that confirm the existence of the dynamic–convective sublayer. A more direct verification of the conditions implying the existence of this sublayer could also in principle be established by measurements (or determination with direct numerical simulation or the LES method) of the moments $\langle p \, \partial u/\partial x \rangle$, $\langle p \, \partial v/\partial y \rangle$ and $\langle p \, \partial w/\partial z \rangle$ describing the rates of energy exchanges among the velocity components. Let us hope that such a verification will become possible in the near future.

3. Mean velocity and temperature profiles in unstable boundary layers

In the dynamic sublayer where buoyancy effects can be neglected the well-known logarithmic formulae for $U(z)$ and $T(z)$ are valid. In the case of a rough wall the formulae have the form

$$U(z) = \frac{u_*}{\kappa} \ln \frac{z}{z_U}, \quad T(z) = T_\mathrm{w} - \frac{T_* P_\mathrm{t}}{\kappa} \ln \frac{z}{z_T}, \tag{3.1}$$

where T_w is the wall temperature and z_U and z_T are roughness parameters for the velocity and temperature. (Note that both experimental data and model theoretical estimates show that z_T is much smaller than z_U; see e.g. Yaglom 1974, 1979 and Kader & Yaglom 1980.)

At heights z of the order of L_* the profiles $U(z)$ and $T(z)$ begin to deviate from the logarithmic shapes, and at greater heights the dynamic–convective sublayer is established where the horizontal and vertical velocity fluctuations are energetically uncoupled. In this sublayer, directional dimensional analysis with different dimensions L_x and L_z of horizontal and vertical lengths can be applied to the study of the turbulent regime. Therefore the velocity scale u_* of dimension $L_x^{\frac{1}{2}} L_z^{\frac{1}{2}} T_\mathrm{I}^{-1}$ is inconvenient here and it must be replaced by two scales: a vertical velocity scale $w_* = (Q\beta z)^{\frac{1}{3}}$ (of dimension $L_z T_\mathrm{I}^{-1}$) and a horizontal velocity scale $u_{**} = u_*^2/w_* =$

$u_*^2(Q\beta z)^{-\frac{1}{3}}$ (of dimension $L_x T_1^{-1}$). The temperature scale $T_* = Q/u_*$ is also inconvenient here and it must be replaced by $\Theta_* = Q/w_* = Q^{\frac{2}{3}}(\beta z)^{-\frac{1}{3}}$ (of the same dimension Θ_1 as the temperature). Since in the framework of the directional analysis no combination of Q, β and u_* has the same dimension as the vertical coordinate z, the mean velocity and temperature profiles in the dynamic–convective sublayer must be self-preserving and described by the equations

$$\frac{\mathrm{d}U}{\mathrm{d}z} = B_U \frac{u_{**}}{z} = B_U u_*^2 (Q\beta)^{-\frac{1}{3}} z^{-\frac{4}{3}}, \quad \frac{\mathrm{d}T}{\mathrm{d}z} = -B_T \frac{\Theta_*}{z} = -B_T Q^{\frac{2}{3}} \beta^{-\frac{1}{3}} z^{-\frac{4}{3}}, \tag{3.2a}$$

where B_U and B_T are dimensionless constants. Note that (3.2*a*) have the form proposed by Prandtl (1932) and Obukhov (1946) who assumed, however, that these equations relate to the limiting case of very strong instability (cf. (1.5) and (1.6)).

The upper edge of the dynamic–convective sublayer is at height z_1 of the order of L. Beyond this sublayer the profiles $U(z)$ and $T(z)$ begin to deviate from the shapes given by (3.2*a*). At even greater heights the free convection sublayer is established where the value of the friction velocity does not affect the statistical regime. Here $w_* = (Q\beta z)^{\frac{1}{3}}$ is the only relevant velocity scale and $\Theta_* = Q/w_*$ is the relevant temperature scale; hence in the free convection sublayer

$$\frac{\mathrm{d}U}{\mathrm{d}z} = C_U \frac{w_*}{z} = C_U (Q\beta)^{\frac{1}{3}} z^{-\frac{2}{3}}, \quad \frac{\mathrm{d}T}{\mathrm{d}z} = -C_T \frac{\Theta_*}{z} = -C_T Q^{\frac{2}{3}} \beta^{-\frac{1}{3}} z^{-\frac{4}{3}}, \tag{3.3a}$$

where C_U and C_T are two new constants. Equations (3.2*a*) and (3.3*a*) for $\mathrm{d}T/\mathrm{d}z$ are of the same form but the coefficients B_T and C_T can differ from each other. According to (1.3)

$$\phi_U(\zeta) = B_U \zeta^{-\frac{1}{3}}, \quad \phi_T(\zeta) = B_T \zeta^{-\frac{1}{3}} \tag{3.2b}$$

in the dynamic–convective sublayer and

$$\phi_U(\zeta) = C_U \zeta^{\frac{1}{3}}, \quad \phi_T(\zeta) = C_T \zeta^{-\frac{1}{3}} \tag{3.3b}$$

in the free convection sublayer.

In the framework of the directional analysis the coefficient C_U is not a dimensionless constant but a parameter of dimension L_x/L_z. By adding C_U (or, equivalently, the quantity $\kappa_1 = C_U^{-\frac{1}{2}}$ of dimension $(L_z/L_x)^{\frac{1}{2}}$) to the list of relevant parameters we can apply directional analysis to the study of turbulence in the flow region above the dynamic–convective sublayer. Then we can introduce one more vertical length, $L_{**} = u_*^3/C_U^{\frac{3}{2}} Q\beta = (\kappa_1 u_*)^3/Q\beta$; the lower edge of the free convection sublayer could be expected to be at heights of the order of L_{**}.

Let us now consider the comparison of equations (3.1) (or, equivalently, (1.3) and (1.4)), (3.2*a*) and (3.3*a*) with the available experimental data. The formulae (3.1) and (1.3)–(1.4) relate to unstratified boundary layers; they are confirmed by extensive laboratory and atmospheric data. All the data show that $\kappa \approx 0.4$ (see, for example, recent meteorological papers by Högström 1985, Telford *et al.* 1986 and Zhang, Oncley & Businger 1988). Reliable measurements of P_t have only been performed in laboratories; they show that $P_t \approx 0.85$ and this estimate does not contradict the available atmospheric data (see, for example, Yaglom 1974, 1979 and Kader & Yaglom 1980).

It was discovered in the late 1960s and early 1970s that equations (3.2*a*) agree satisfactorily with wind velocity and temperature observations in unstable surface layers at heights in the range from approximately $z = 0.04|L_O| = 0.1L$ to $z = |L_O| = 2.5L$, but at greater heights the experimental values of $\phi_U(\zeta)$ deviate upward from the curve $\phi_U = B_U \zeta^{-\frac{1}{3}}$ and the experimental values of $\phi_T(\zeta)$ deviate downward from $\phi_T = B_T \zeta^{-\frac{1}{3}}$ (see e.g. Monin & Yaglom 1971 and Businger *et al.* 1971). Because of the indicated deviation of the experimental data from equation (3.2*a*), the dimensionless temperature and velocity profiles in unstable thermal stratification have often been described by the following empirical equations:

$$\phi_U(\zeta) = A_U(1-\gamma_U \zeta)^{-\frac{1}{4}}, \quad \phi_T(\zeta) = A_T(1-\gamma_T \zeta)^{-\frac{1}{2}}, \tag{3.4}$$

where $A_U, A_T, \gamma_U, \gamma_T$ are four dimensionless coefficients (see e.g. Dyer 1967; Businger *et al.* 1971; Francey & Garratt 1981; Dyer & Bradley 1982 and Högström 1988). According to these equations $\phi_U \propto \zeta^{-\frac{1}{4}}$ and $\phi_T \propto \zeta^{-\frac{1}{2}}$ for $\zeta \gg 1$. However, the values of coefficients A_U, A_T and, especially, γ_U, γ_T that were used by various authors differ substantially and in fact all the suggested equations of the form (3.4) were based mostly on data relating only to $z < 2.5|L_0| \approx 6L$. Therefore the results of the cited papers are insufficient for reasonably reliable determination of the asymptotic behaviour of profiles for very large values of $\zeta = z/L$.

The search for more complete data covering a wider range of ζ-values was an important reason to arrange detailed micrometeorological measurements at the Tsimlyansk Field Station of the Moscow Institute of Atmospheric Physics. The measurement site is a flat and rather homogeneous area in southern Russia where in the summers of 1981–1987 extensive data were collected. The data included many mean wind and temperature profiles determined by cup anemometers and resistance thermometers placed at 1, 2, 4, 8, 16, 24, 32 and 40 m on a 40 m mast. Simultaneously with the profile measurements, eddy correlation measurements of $u_* = \langle -uw \rangle^{\frac{1}{2}}$ and $Q = \langle wt \rangle$ were performed with the aid of a three-component sonic anemometer and a fast-response platinum resistance thermometer placed close to each other at one height (which often varied) within the surface layer. The measurement technique and the data treatment methods are described in detail by Kader & Perepelkin (1984) and only a brief outline of them will be given here. It is worth noting, however, that during 1985–1987 much attention was given to the conditions of hot sunny weather and weak but stable wind which correspond to especially small values of L. As a result the data were collected for a wide range of positive ζ-values up to $\zeta = z/L \approx 60$ (i.e. $|\zeta_0| = z/|L_0| \approx 25$).

The measured values of the wind velocity and temperature at eight heights were averaged over 35-minute time intervals and the mean profiles $U(z)$ and $T(z)$ obtained were then approximated by a special smoothing function containing four empirical coefficients determined by the least-squares method. The deviations of the measured values of $U(z)$ and $T(z)$ from the approximating curves were always within the range of the measurement errors. Therefore the determination of the gradients $\mathrm{d}U/\mathrm{d}z$ and $\mathrm{d}T/\mathrm{d}z$ from the curves seemed to be justified. The gradients were computed for six heights (the lowest height (1 m) and the upper height (40 m) were excluded since the procedure described was not accurate enough there). The value of L was determined from the measured values of u_* and Q, and the values of $\phi_U(\zeta)$ and $\phi_T(\zeta)$ were computed for six heights z. All the values of $\phi_U(\zeta)$ and $\phi_T(\zeta)$ obtained were plotted on graphs together with the values of $\zeta^a \phi_U(\zeta)$ and $\zeta^b \phi_T(\zeta)$ for $a = \frac{1}{3}$ and $-\frac{1}{3}$, $b = \frac{1}{3}$. The values of the exponents a and b were chosen so that the products $\zeta^a \phi_U(\zeta)$ and $\zeta^b \phi_T(\zeta)$ were constant in the ranges of ζ-values where equations (3.2*b*) and (3.3*b*) are valid. Note that $\zeta^{-\frac{1}{3}}\phi_U(\zeta)$ and $\zeta^{\frac{1}{3}}\phi_T(\zeta)$ do not depend on u_*; hence plotting these functions did not produce the 'artificial correlations' introduced by using the dependent and independent variables which both include the experimental values of u_* subject to considerable measurement errors (see Hicks 1978, 1981). The Tsimlyansk data are

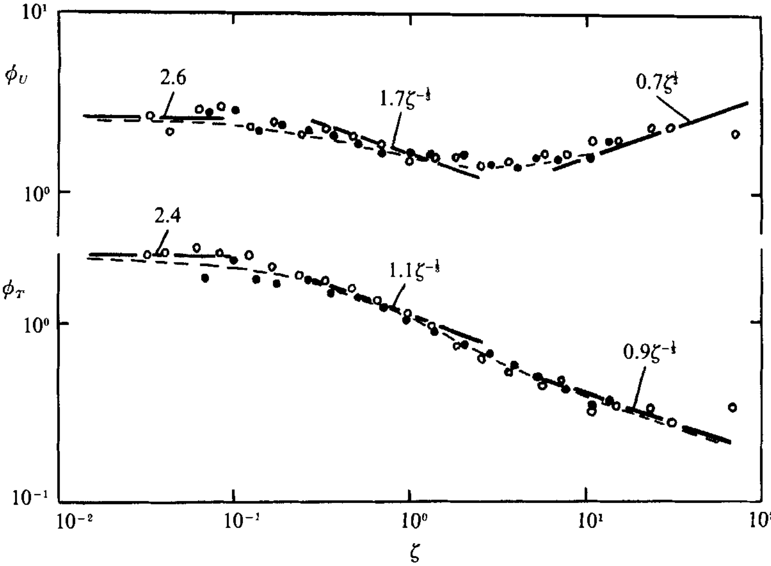

FIGURE 1. Dimensionless velocity and temperature gradients $\phi_U(\zeta)$ and $\phi_T(\zeta)$ obtained from the Tsimlyansk data of 1981–87 (open circles) and from the data collected in Kansas in 1968, Minnesota, 1973, and Australia, 1976, (closed circles); ——, equations (3.5); – – –, (3.6).

rather scattered (as are all the data of micrometeorological measurements), but they are very numerous and this reduces the influence of the scatter. In the course of data processing, the ζ-axis was partitioned into a number of comparatively narrow subintervals and the mean values of the dependent variable were computed for each of the subintervals. The obtained mean values were found to agree quite satisfactorily with the assumption on the constancy of $\phi_U(\zeta)$ and $\phi_T(\zeta)$ in a range of small ζ-values, the constancy of $\zeta^{\frac{1}{3}}\phi_U(\zeta)$ and $\zeta^{\frac{1}{3}}\phi_T(\zeta)$ in a range of moderate ζ-values and the constancy of $\zeta^{-\frac{1}{3}}\phi_U(\zeta)$ and $\zeta^{\frac{1}{3}}\phi_T(\zeta)$ in a range of large ζ-values. Namely, according to these data

$$\phi_U(\zeta) = \begin{cases} 2.6 & \text{for} \quad 0 < \zeta \lesssim 0.1, \\ 1.7\zeta^{-\frac{1}{3}} & \text{for} \quad 0.3 \lesssim \zeta \lesssim 3, \\ 0.7\zeta^{\frac{1}{3}} & \text{for} \quad \zeta \gtrsim 5, \end{cases} \quad \phi_T(\zeta) = \begin{cases} 2.4 & \text{for} \quad 0 < \zeta \lesssim 0.1, \\ 1.1\zeta^{-\frac{1}{3}} & \text{for} \quad 0.3 \lesssim \zeta \lesssim 3, \\ 0.9\zeta^{-\frac{1}{3}} & \text{for} \quad \zeta \gtrsim 5 \end{cases} \tag{3.5}$$

(see figure 1). Hence the data agree satisfactorily with the assumption on the existence of dynamic, dynamic–convective and free convection sublayers of an unstable boundary layer and show that these sublayers are located approximately at heights $z \lesssim 0.1L = 0.04|L_O|$, $0.12|L_O| = 0.3L \lesssim z \lesssim 3L = 1.2|L_O|$ and $z \gtrsim 5L = 2|L_O|$; moreover, $A_U \approx 2.6$, $A_T \approx 2.4$, $B_U \approx 1.7$, $B_T \approx 1.1$ and $C_U \approx 0.7$, $C_T \approx 0.9$. The (rather crude) estimate of C_U obtained implies that $L_{**} \approx 2L$. Note that in the range $0.2 \lesssim \zeta \lesssim 2$ the data shown in figure 1 can also be approximated quite accurately by equations of the form (3.4). However, such equations have no theoretical justification and they are clearly inadequate in the range of large values of ζ.

The estimates of A_U and A_T given agree well with the results of numerous laboratory measurements showing that $A_U \approx 2.5$, $A_T/A_U = P_t \approx 0.85$. The values of B_U and B_T obtained are within the scatter of the previous estimates of these coefficients from meteorological measurements which were collected by Monin & Yaglom (1971, §8). It was, however, suggested in the book by Monin & Yaglom that the corresponding estimates of B_U and B_T are related to a hypothetical 'free

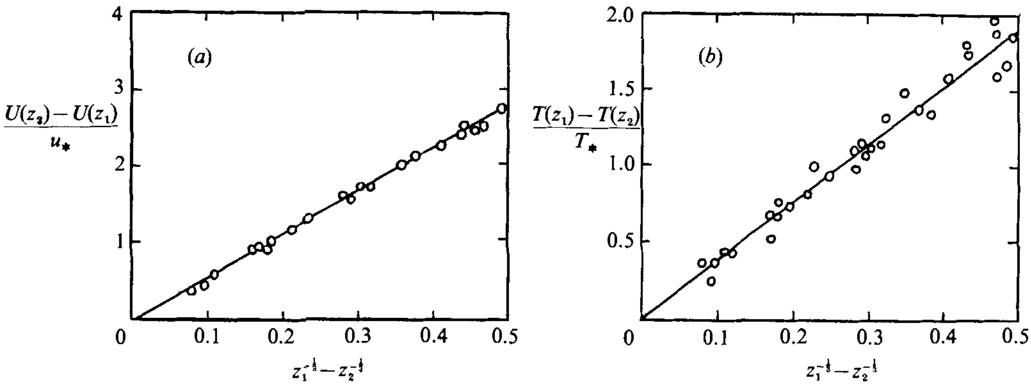

FIGURE 2. Mean profiles in the laboratory unstably stratified flow from Petukhov & Polyakov (1988): (*a*) velocity profile; (*b*) temperature profile.

convection region' which occupies the sublayer from $z \approx 0.1L$ to the upper edge of the ASL. Figure 1 also shows that equation (3.2*b*) for $\phi_T(\zeta)$ is in fact valid with a good accuracy starting at $\zeta = 0.1$, but (3.2*b*) for $\phi_U(\zeta)$ is clearly invalid at such small values of ζ. Moreover, although the coefficients B_T and C_T differ only slightly, the values of $\phi_T(\zeta)$ for $\zeta > 3$ are clearly below the curve $\phi_T = 1.1\zeta^{-\frac{1}{3}}$. As to the data from laboratory experiments, only the results by Petukhov & Polyakov (1988, §5.1), obtained in a plane channel with a heated lower wall are appropriate for comparison with the theoretical equations (3.2*a*). In figure 2 the dependence of $[U(z_2)-U(z_1)]/u_*$ and $[T(z_1)-T(z_2)]/T_*$ on $[z_1^{-\frac{1}{3}}-z_2^{-\frac{1}{3}}]$ are shown, which should be linear with slopes $3B_U$ and $3B_T$, respectively, if equations (3.2*a*) are valid. The data in figure 2 are for the flow region which was called the 'strong convection region' by the authors, but, in fact, it belongs to the dynamic–convective sublayer where equations (3.2*a*) must hold. According to figure 2, $3B_U \approx 5.6$, $3B_T \approx 3.8$, i.e. $B_U \approx 1.9$, $B_T \approx 1.3$; these estimates are quite close to those in equations (3.5).

The values of the coefficients C_U and C_T in (3.3*a*) have never been estimated before. Note in this respect that in all the previous descriptions of free convection (e.g. by Obukhov 1960 and Wyngaard *et al.* 1971) it has been assumed that the wind shear vanishes there and the turbulence becomes axisymmetric. Hence it has been implicitly supposed that $C_U = 0$. It has also already been noted in §2 that Betchov & Yaglom (1971), Yaglom (1974), Monin & Yaglom (1975) and Wyngaard *et al.* (1971) even suggested that the free convection layer, where the influence of u_* can be neglected, possibly cannot be attained in the atmosphere. However, according to the data analysed in this paper, the free convection sublayer can quite often be observed in the atmosphere, but the wind shear differs from zero (but it is small) there and the turbulence is not axisymmetric. (This does not contradict the absence of substantial influence of u_* on the flow in the free convection region.) Therefore the free convection sublayer considered in this paper is a generalization of the free convection regime introduced in the earlier works.

Apparently, even the non-monotonicity of the function $\phi_U(\zeta)$ has never been noticed before. However, when the numerous data of field experiments in Kansas in 1968, Minnesota, 1973, and Australia, 1976, published by Izumi (1971), Izumi & Caughey (1976), Garratt *et al.* (1979) and Dyer, Garratt & Francey (1981), were treated similarly to the Tsimlyansk data, it was found that all these data agree satisfactorily with equations (3.5) (see figure 1). This agreement seems surprising, since the data from these three experiments have been used previously to obtain rather different conclusions; see Businger *et al.* (1971), Francey & Garratt (1981), Dyer & Bradley (1982) and Högström (1988). Note, however, that no figures in these papers include any data for $\zeta > 8$ and the majority of the figures are based on measurements related to an even more restricted range of positive ζ-values. Possibly the rare observations at larger values of ζ, which can be found in the published collections of numerical data, were considered as unreliable and were not taken into account in data treatments whose results were described in the four above-mentioned papers. It seems to us, however, that the extensiveness of the data for large values of ζ collected in all the experiments taken together and the fact that the results obtained for four very different geographical areas proved to be close to each other must give the data great credibility.

The behaviour of the functions $\phi_U(\zeta)$ and $\phi_T(\zeta)$ in the transition regions between the dynamic and dynamic–convective, or dynamic–convective and free convection sublayers cannot be found from dimensional reasoning. Therefore here one should apply some additional hypotheses to compute these functions or should limit oneself to finding purely empirical formulae for them. Such purely empirical formulae of the form

$$\phi_U(\zeta) = 2.5\left[\frac{1+0.1\zeta^2}{1+3\zeta}\right]^{\frac{1}{3}}, \quad \phi_T(\zeta) = 1.6\left[\frac{3+\zeta}{1+4\zeta+8\zeta^2}\right]^{\frac{1}{3}} \tag{3.6}$$

were proposed by Kader & Perepelkin (1989). The curves described by (3.6) are quite smooth and have the correct asymptotic behaviour at both small and large values of ζ; moreover, as figure 1 shows, they agree rather well with the data for all the values of ζ investigated.

4. Profiles of one-point fluctuation moments

4.1. *Second-order moments*

Let us now consider the moments of the turbulent fluctuations in unstably stratified boundary layers and begin with the second-order moments. There are only five non-vanishing second-order moments that vary with z; they are $\langle u^2\rangle = \sigma_u^2$, $\langle v^2\rangle = \sigma_v^2$, $\langle w^2\rangle = \sigma_w^2$, $\langle t^2\rangle = \sigma_t^2$ and $\langle ut\rangle = -Q_x$. It is known that within the dynamic sublayer all the moments take constant values which depend only on u_* and $Q = \langle wt\rangle$. In particular,

$$\sigma_u = A_1 u_*, \quad \sigma_v = A_2 u_*, \quad \sigma_w = A_3 u_*, \quad \sigma_t = A_4 T_*, \quad Q_x = A_5 Q, \tag{4.1}$$

where $A_1, \ldots, A_5$ are dimensionless constants. In the dynamic–convective sublayer the vertical velocity scale $w_* = (Q\beta z)^{\frac{1}{3}}$, horizontal velocity scale $u_{**} = u_*^2/w_*$ and temperature scale $\Theta_* = Q/w_*$ must be used; therefore

$$\left.\begin{aligned} &\sigma_u = B_1 u_{**} = B_1 u_*^2 (Q\beta z)^{-\frac{1}{3}}, \quad \sigma_v = B_2 u_*^2 (Q\beta z)^{-\frac{1}{3}}, \quad \sigma_w = B_3 w_* = B_3 (Q\beta z)^{\frac{1}{3}}, \\ &\sigma_t = B_4 \Theta_* = B_4 Q^{\frac{2}{3}}(\beta z)^{-\frac{1}{3}}, \quad Q_* = B_5 u_{**}\Theta_* = B_5 u_*^2 Q^{\frac{1}{3}}(\beta z)^{-\frac{2}{3}} \end{aligned}\right\} \tag{4.2}$$

in this sublayer. Finally, in the free convection sublayer the parameter u_* is unimportant and w_* and Θ_* are the only relevant velocity and temperature scales. Therefore equations (1.9) must be valid in this sublayer.

Equations (4.1) are well known and have been verified many times in laboratory experiments. The available laboratory data agree well with the equations and show that

$$A_1 \approx 2.3, \quad A_2 \approx 1.7, \quad A_3 \approx 1.0, \quad A_4 \approx 1.3, \quad A_5 \approx 2.5 \tag{4.3a}$$

(see e.g. Yaglom 1979 and Kader & Yaglom 1980). Many atmospheric values of all the second-order turbulence moments were computed at the Tsimlyansk field station by the time averaging (over 35-minute intervals) of the squares (or the products) of fluctuations measured by the fast-response instruments (sonic anemometer and platinum-wire thermometer). The results for neutral thermal stratification are more scattered than most of the laboratory data; however, on the whole they also agree well with (4.1) and give the following estimates for the coefficients:

$$A_1 \approx 2.7, \quad A_2 \approx 2.5, \quad A_3 \approx 1.25, \quad A_4 \approx 2.9, \quad A_5 \approx 3.8. \qquad (4.3b)$$

The estimates (4.3*b*) for A_1, A_2 and A_3 are slightly larger than the corresponding estimates (4.3*a*) but the differences are not great and can be explained by the experimental errors. (The unexpectedly high estimate (4.3*b*) of A_2 is apparently due to wind-direction fluctuations which are absent in the laboratory flows.) Let us also note that available atmospheric estimates of A_1, A_2 and A_3 in the literature are usually intermediate between (4.3*a*) and (4.3*b*); for example, according to Kerman (1978) $A_1 \approx 2.5, A_2 \approx 1.8, A_3 \approx 1.16$; Yasuda (1978) found that $A_1 \approx 2.35, A_2 \approx 1.9, A_3 \approx 1.0$; Binkowski (1979) recommended the estimates $A_1 \approx 2.5, A_2 \approx 1.9$ and $A_3 \approx 1.25$; while Bradley & Antonia (1979) came to the conclusion that $A_1^2+A_2^2+A_3^2 = 12.5$. The latter authors noted that most of the laboratory estimates of A_1, A_2 and A_3 are slightly lower than the same estimates based on atmospheric data and they discussed possible reasons of such a discrepancy.

The situation concerning the coefficients A_4 and A_5 is more complicated since the differences between their estimates (4.3*a*) and (4.3*b*) exceed possible experimental errors. These differences show that temperature fluctuations in an atmospheric surface layer at neutral (or almost neutral) thermal stratification are larger than those in the logarithmic region of a laboratory boundary layer. It is natural to think that this fact is due to the thermal inhomogeneities that always exist on the ground while in laboratory flows with heat transfer the wall is usually thermally homogeneous. Note in this respect that in the laboratory boundary layers without heat transfer (i.e. when the wall and fluid have the same temperature), there are no temperature fluctuations at all, while in an atmospheric surface layer such fluctuations exist even if the mean heat flux Q is equal to zero (Yasuda 1978).

Until now there have been no reliable data on the values of $\langle u^2\rangle$ and $\langle v^2\rangle$ in the dynamic–convective and free convection sublayers of an unstable ASL. The heights z for these sublayers are greater than for the dynamic sublayer and the increase of z leads to a shift of energy ranges (i.e. ranges which give the main contribution to the mean square values) in the spectra of horizontal wind components towards the low-frequency end of the frequency axis. A similar spectral shift is produced by the instability intensification (i.e. decrease of L). Therefore, when the wind fluctuations are measured in the ASL above the dynamic sublayer, the band-pass frequency of the instrument used (i.e. the band of frequencies of the oscillations that are not distorted by the instrument) proves to be usually insufficient for obtaining accurate values of $\langle u^2\rangle$ and $\langle v^2\rangle$. Note in this connection that the atmospheric data on $\langle u^2\rangle$ and $\langle v^2\rangle$ analysed by Panofsky *et al.* (1977) demonstrate the dependence of these moments on the thickness z_i of the planetary boundary layer and z_i is always much greater than the thickness of the ASL. (In summer at midday z_i is usually of the order of 1000 m.) The conclusions drawn by Panofsky *et al.* do not seem to be very reliable when applied to the ASL; they are based mostly on measurements at relatively large heights (of the order of several tens or even hundreds of metres) and their validity for smaller heights of the order of a few metres needs additional verification.

However, if the values of $\langle u^2 \rangle$ and $\langle v^2 \rangle$ in the ASL really depend on z_i, then this means that the spectra of the corresponding fluctuations are extended to the range of very small frequencies where the Monin–Obukhov similarity theory is inapplicable. It seems to us that till now there have been no reliable measurements of $\langle u^2 \rangle$ and $\langle v^2 \rangle$ above the dynamic sublayer of the ASL; therefore no experimental data on $\langle u^2 \rangle$ and $\langle v^2 \rangle$ are considered in this paper.

It has been noted above that it was known as early as the 1960s that the relationships $\sigma_w \propto z^{\frac{1}{3}}$ and $\sigma_t \propto z^{-\frac{1}{3}}$ agree with the available data over a wide range of heights z exceeding $0.1L$. In subsequent years the measurements of $\langle w^2 \rangle$ and $\langle t^2 \rangle$ in the ASL were repeated by many experimenters who came to the same conclusion; see e.g. Wyngaard *et al.* (1971), Monji (1973), Businger (1973) and Panofsky *et al.* (1977). However, the theory presented in this paper casts doubt on the available estimates of the dimensionless proportionality coefficients in these relationships since most of these estimates are based on data for the height range that includes not only the dynamic–convective sublayer but also a part of (or all) the transitional region and sometimes also a part of the free convection sublayer. (Let us remind readers in this respect that equations for profiles of σ_w and σ_t in the dynamic–convective and free convection sublayers have the same form, but the numerical coefficients in these equations may have different values. If these values for two ranges differ only slightly then it is possible, of course, to approximate the whole profile by one equation with an intermediate value of the coefficient.) Finally, the applicability of equation (4.2) for the profile of Q_x to a range of z-values larger than a few tenths of L was demonstrated by Zilitinkevich (1971) who used the earliest measurements of the moment Q_x.

More complete results, which included numerous values of the second-order moments over a wide range of ζ-values, were obtained at the Tsimlyansk station during 1981–1987. The Tsimlyansk data are shown in figure 3; they agree quite satisfactorily with equations (4.1), (4.2) and (1.9) for σ_w, σ_t and Q_x and give the following estimates of the numerical coefficients in (4.2) and (1.9):

$$B_3 \approx 1.65, \quad B_4 \approx 1.4, \quad B_5 \approx 1.2; \quad C_3 \approx 1.3, \quad C_4 \approx 1.5. \tag{4.4}$$

A similar graph is plotted in figure 4 for the data from a number of other papers (Wyngaard *et al.* 1971; Mordukhovich & Tsvang 1966; Zubkovskii & Tsvang 1966; Haugen, Kaimal & Bradley 1971; Arya 1972; Hayashi 1974; Monji 1973, 1975; Donelan & Miyake 1973; Wesely 1974; Volkov, Koprov & Kravchenko 1975; Rayment & Caughey 1977; Bradley, Antonia & Chambers 1981*a*, *b*, 1982; Schacher *et al.* 1981; Kai 1982). The data from these sources cover an even wider range of ζ-values extending up to $\zeta \approx 100$; they are rather scattered but do not contradict equations (4.1), (4.2) and (1.9), and lead to almost the same estimates of the dimensionless coefficients as the Tsimlyansk data. It is worth noting in this respect that the most usual previous estimates of the coefficients D_3 and D_4 in the relations $\sigma_w/u_* = D_3|\zeta_O|^{\frac{1}{3}}$, $\sigma_t/T_* = D_4|\zeta_O|^{-\frac{1}{3}}$, where $|\zeta_O| \approx \kappa\zeta \approx 0.4\zeta$, are: $D_3 \approx 2.0$, $D_4 \approx 0.9$ (see for example Wyngaard *et al.* 1971; Monji 1973; Businger 1973; Panofsky *et al.* 1977; Tennekes 1984). Replacing $|\zeta_O|$ by 0.4ζ we obtain the corrected coefficients $D_3' = 1.5$, $D_4' = 1.25$ which, according to the above-mentioned authors, relate to a fictional free convection region which is, in fact, a combination of the dynamic–convective sublayer, the transitional region and the free convection sublayer. We see that the estimate of D_3' is intermediate between those of B_3 and C_3, while the estimate of D_4' is quite close to that of B_4.

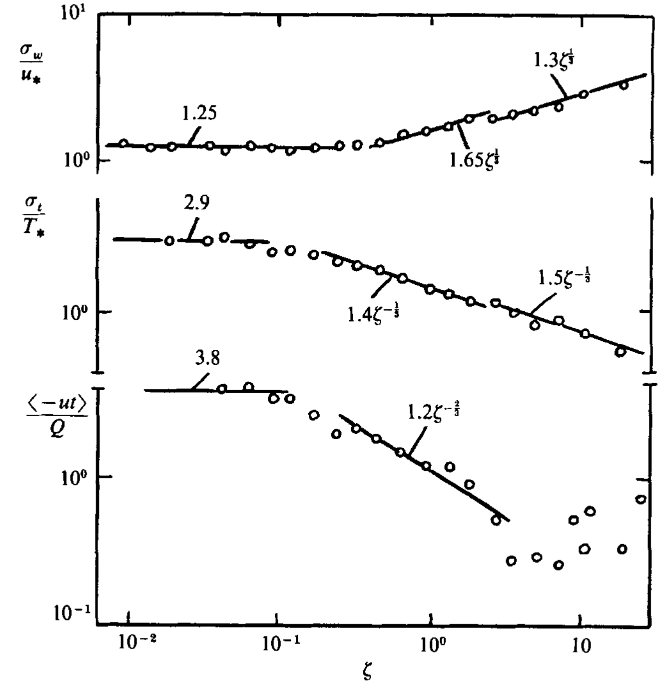

FIGURE 3. Profiles of normalized second-order moments in the unstable surface layer according to the Tsimlyansk data of 1981–87.

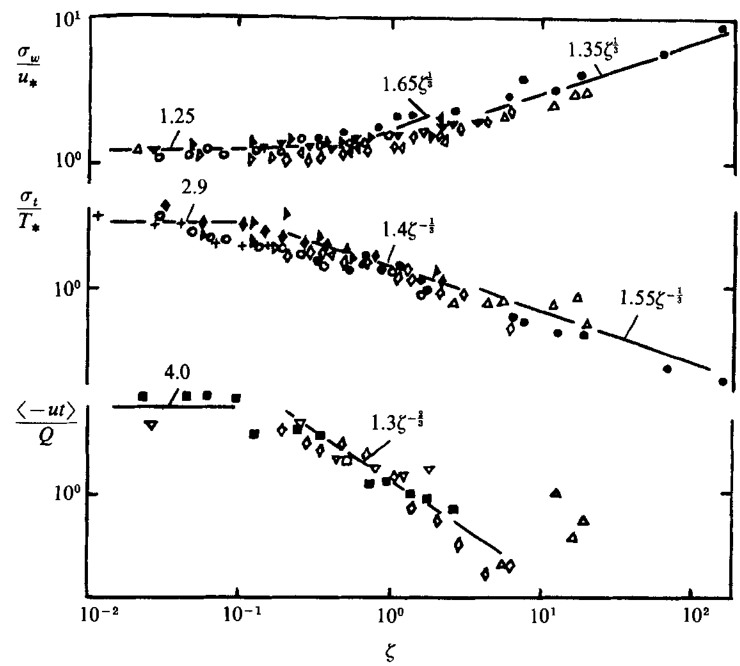

FIGURE 4. The same profiles as in figure 3 according to data from the literature. ○, Mordukhovich & Tsvang (1966); ▽, Zubkovskii & Tsvang (1966); ◇, Wyngaard *et al.* (1971); ▼, Haugen *et al.* (1971); ◆, Arya (1972); ▷, Hayashi (1974); ●, Monji (1973, 1975); ◀, Donelan & Miyake (1973); ◁, Wesely (1974); +, Volkov *et al.* (1975); △, Rayment & Caughey (1977); ■, Bradley *et al.* (1981*a*, *b*, 1982); □, Schacher *et al.* (1981); ▶, Kai (1982).

4.2. *Third-order moments*

Results similar to (4.1), (4.2) and (1.9) can also be given for third-order moments of turbulent fluctuations. We shall consider here only the moments $\langle w^2t\rangle$, $\langle wt^2\rangle$, $\langle uwt\rangle$, $\langle uw^2\rangle$, $\langle ut^2\rangle$ and $\langle t^3\rangle$ since only these third-order moments were computed from the Tsimlyansk data. (The third-order moments which include u^2 or v^2 were not considered since the results of their estimation seemed to be unreliable. The values of $\langle w^3\rangle$ were not computed, by mistake; a survey of available experimental estimates of the values of $\langle w^3\rangle$ in the ASL was published by Chiba (1978) and it shows that these estimates are very scattered but on the whole they do not contradict the theory presented in this paper.) In the dynamic sublayer the third-order moments measured in Tsimlyansk must be independent of z and have the form

$$\left.\begin{aligned}&\langle w^2t\rangle = A_6 u_*^2 T_* = A_6 u_* Q, \quad \langle wt^2\rangle = A_7 Q^2/u_*, \quad \langle uwt\rangle = -A_8 u_* Q,\\ &\langle uw^2\rangle = -A_9 u_*^3, \quad \langle ut^2\rangle = -A_{10} Q/u_*, \quad \langle t^3\rangle = A_{11} Q^3/u_*^3,\end{aligned}\right\} \tag{4.5}$$

where $A_6, \ldots, A_{11}$ are some constants. Similarly, within the dynamic–convective sublayer the following equations must be valid:

$$\left.\begin{aligned}&\langle w^2t\rangle = B_6 Q^{\frac{4}{3}}(\beta z)^{\frac{1}{3}}, \quad \langle wt^2\rangle = B_7 Q^{\frac{5}{3}}(\beta z)^{-\frac{1}{3}}, \quad \langle uwt\rangle = -B_8 u_*^2 Q^{\frac{2}{3}}(\beta z)^{-\frac{1}{3}},\\ &\langle uw^2\rangle = -B_9 u_*^2 (Q\beta z)^{\frac{1}{3}}, \quad \langle ut^2\rangle = -B_{10} u_*^2 Q(\beta z)^{-1}, \quad \langle t^3\rangle = B_{11} Q^2(\beta z)^{-1}.\end{aligned}\right\} \tag{4.6}$$

Finally, within the free convection sublayer

$$\left.\begin{aligned}&\langle w^2t\rangle = C_6 Q^{\frac{4}{3}}(\beta z)^{\frac{1}{3}}, \quad \langle wt^2\rangle = C_7 Q^{\frac{5}{3}}(\beta z)^{-\frac{1}{3}}, \quad \langle uwt\rangle = -C_8 Q^{\frac{4}{3}}(\beta z)^{\frac{1}{3}},\\ &\langle uw^2\rangle = -C_9 Q\beta z, \quad \langle ut^2\rangle = -C_{10} Q^{\frac{5}{3}}(\beta z)^{-\frac{1}{3}}, \quad \langle t^3\rangle = C_{11} Q^2(\beta z)^{-1}.\end{aligned}\right\} \tag{4.7}$$

It is known that the third-order moments of unordered fluctuations are always considerably less accurately determined from a given fluctuation record than the second-order ones (i.e. much longer records are needed to achieve the same accuracy for the third moments). Therefore it is not surprising that all the experimental data related to the third moments of turbulent fluctuations are very scattered. Figure 5 shows, however, that none of the available results (both calculated from the Tsimlyansk measurements and borrowed from the available literary sources) contradict the theoretical equations (4.5)–(4.7), and the data allow one to give the following (rather rough) estimates of some coefficients in these equations:

$$\left.\begin{aligned}&A_6 \approx 0.55, \quad A_7 \approx 1.2, \quad A_{10} \approx 4.5, \quad A_{11} \approx 10;\\ &B_6 \approx 1.0, \quad B_7 \approx 1.1, \quad B_8 \approx 1.0, \quad B_9 \approx 0.8, \quad B_{10} \approx 1.0, \quad B_{11} \approx 2;\\ &C_7 \approx 1.6, \quad C_8 \approx 0.45, \quad C_9 \approx 0.35, \quad C_{10} \approx 0.75.\end{aligned}\right\} \tag{4.8}$$

Of course, these estimates must be considered as preliminary guesses to be revised on the basis of subsequent special measurements.

The results of the early atmospheric measurements of the moment $\langle uwt\rangle$ were published by Wyngaard *et al.* (1971) who noted that the data for unstable stratification disagree strongly with the simplest free convection prediction (4.7) for this moment. It was stressed by Betchov & Yaglom (1971) that the data by Wyngaard *et al.* at $\zeta > 0.5$ agree satisfactorily with (4.6) for the moment $\langle uwt\rangle$. Later Zilitinkevich (1973) used the same data to obtain the first estimate, $B_8 \approx 0.7$, of the coefficient B_8. The moment $\langle w^2t\rangle$ was also measured by Wyngaard *et al.* (1971). These authors concluded that the data obtained agreed well with the free convection

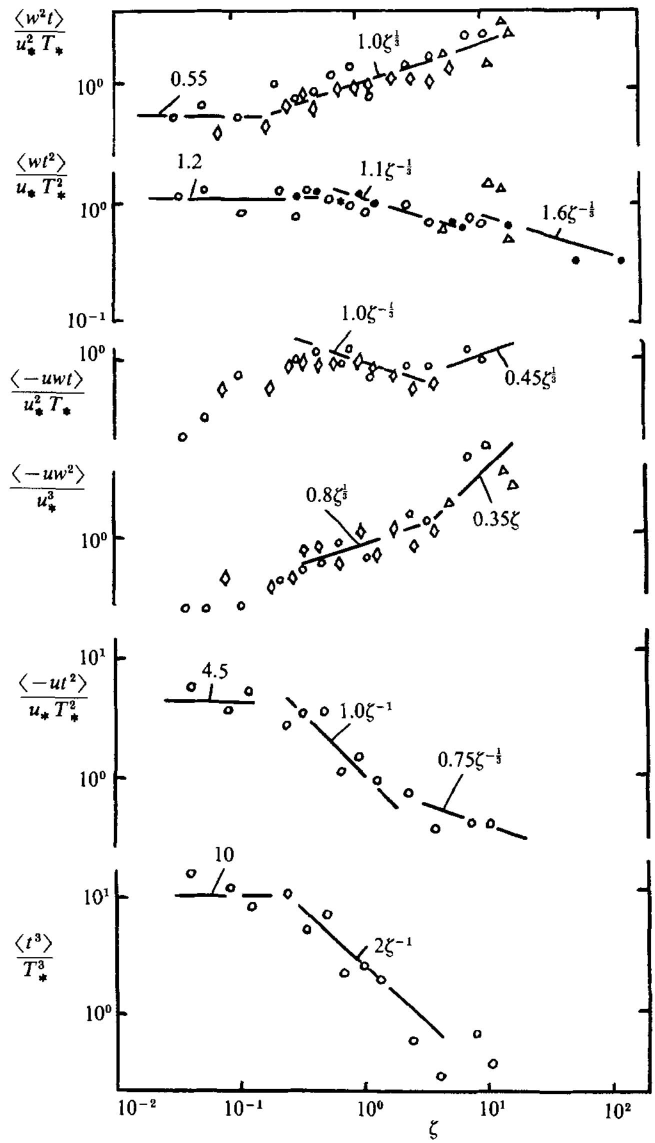

FIGURE 5. Profiles of normalized third-order moments in the unstable surface layer. ○, from the data collected in Tsimlyansk; other symbols have the same meaning as in figure 4.

prediction (4.6) and (4.7) (for $\langle w^2t\rangle$ both these equations have the same form) and gave a rough estimate of the corresponding dimensionless coefficient which implies that $B_6 \approx 0.96$. All the other coefficients in (4.5)–(4.7) have apparently never been estimated before.

4.3. *Budget equations and dissipation rates*

The third-order moments enter the dynamic equations for the second moments. These equations are often called the budget (or balance) equations for the corresponding moments; see e.g. Wyngaard & Coté (1971), Wyngaard *et al.* (1971), Bradley *et al.* (1981, 1982). Thus, the kinetic energy budget in the constant-flux region of a steady two-dimensional boundary layer is the sum of the three equations (2.2); when molecular transfer is neglected, it has the form

$$u_*^2 \frac{\mathrm{d}U}{\mathrm{d}z} + \beta Q - \frac{1}{2}\frac{\mathrm{d}\langle (u^2+v^2+w^2)\,w\rangle}{\mathrm{d}z} - \frac{1}{\rho}\frac{\mathrm{d}\langle pw\rangle}{\mathrm{d}z} = \epsilon, \tag{4.9}$$

where ϵ is the rate of energy dissipation. This equation includes the third moments $\langle wu^2\rangle$, $\langle wv^2\rangle$ and $\langle w^3\rangle$ which determine (together with the moment $\langle pw\rangle$) the rate of the vertical transfer of kinetic energy. Under the same conditions the budget of temperature fluctuation variance $\langle t^2\rangle$ has the form

$$-2Q\frac{\mathrm{d}T}{\mathrm{d}z} - \frac{\mathrm{d}\langle wt^2\rangle}{\mathrm{d}z} = 2N, \tag{4.10}$$

where $N = D\sum_{i=1}^{3}\langle(\partial t/\partial x_i)^2\rangle$ is the dissipation rate for $\frac{1}{2}\langle t^2\rangle$.

The definitions of the dissipation rates ϵ and N include the molecular transfer coefficients ν and D. However, none of the other terms in (4.9) and (4.10) depends on these coefficients and their vertical profiles satisfy the Monin–Obukhov similarity theory; therefore this theory can also be applied to ϵ and N. (Note in this respect that according to the theory of locally isotropic turbulence, ϵ and N are equal to the rates of spectral transfer for the energy, $\frac{1}{2}(\langle u^2\rangle + \langle v^2\rangle + \langle w^2\rangle)$, and temperature fluctuation intensity, $\frac{1}{2}\langle t^2\rangle$; see e.g. Monin & Yaglom (1975, Chap. 8). The rates of spectral transfer are clearly independent of ν and D; this explains the applicability of the Monin–Obukhov theory to ϵ and N.) According to the similarity theory

$$\epsilon = \frac{u_*^3}{z}\phi_\epsilon(\zeta), \quad N = \frac{T_*^2\,u_*}{z}\phi_N(\zeta) = \frac{Q^2}{u_*\,z}\phi_N(\zeta), \tag{4.11}$$

where $\phi_\epsilon(\zeta)$ and $\phi_N(\zeta)$ are two universal functions of ζ.

In the dynamic sublayer the rate of the convective energy production βQ is much smaller than the rate of the dynamic production $u_*^2\,\mathrm{d}U/\mathrm{d}z$; hence the term βQ can be neglected in (4.9). Moreover, none of the one-point moments depends on z in this sublayer; therefore the derivatives of these moments with respect to z are equal here to zero. As a result (4.9) and (4.10) imply that $\epsilon = u_*^2\,\mathrm{d}U/\mathrm{d}z$, $N = -Q\,\mathrm{d}T/\mathrm{d}z$ in the dynamic sublayer and, according to (1.3) and (1.4),

$$\phi_\epsilon(\zeta) = A_U \approx 2.5, \quad \phi(\zeta) = A_T \approx 2.1 \quad \text{for} \quad \zeta \ll 1. \tag{4.12}$$

In the free convection sublayer both ϵ and N must be independent of u_*. Therefore (4.11) implies that

$$\phi_\epsilon(\zeta) = C_\epsilon\,\zeta, \quad \phi_N(\zeta) = C_N\,\zeta^{-\frac{1}{3}} \tag{4.13}$$

in this layer, where C_ϵ and C_N are two constants. By virtue of (4.10), (3.3*a*) and (4.7) we obtain

$$C_N = C_T + \tfrac{1}{6}C_7. \tag{4.14}$$

If z belongs to the dynamic–convective sublayer, then both the terms on the left-hand side of (4.10) are independent of u_*. Hence N is also independent of u_* in the dynamic–convective sublayer and this implies that

$$\phi_N(\zeta) = B_N\,\zeta^{-\frac{1}{3}}. \tag{4.15}$$

According to (4.10), (3.2*a*) and (4.6) we obtain

$$B_N = B_T + \tfrac{1}{6}B_7. \tag{4.16}$$

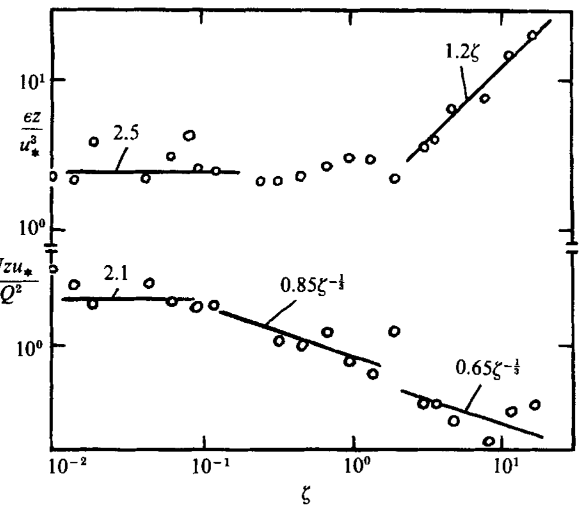

FIGURE 6. Dimensionless profiles of dissipation rates ϵ and N according to the Tsimlyansk data of 1981–87.

The situation for the profile of ϵ in the dynamic–convective sublayer is more complicated. In fact, some of the terms on the left-hand side of (4.9) are proportional to $z^{-\frac{4}{3}}$ in this sublayer, and the others are independent of z. (It is natural to think that the term containing $\langle pw\rangle$ is also the sum of a constant and a term proportional to $z^{-\frac{4}{3}}$.) Therefore

$$\phi_\epsilon(\zeta) = B_\epsilon^{(1)}\zeta + B_\epsilon^{(2)}\zeta^{-\frac{1}{3}} \tag{4.17}$$

in the dynamic–convective sublayer ($B_\epsilon^{(1)}$ and $B_\epsilon^{(2)}$ are unknown constant coefficients).

The dissipation rates ϵ and N can be determined from spectral measurements. According to well-known results by Kolmogorov, Obukhov and Corrsin the following equations are valid in the inertial subrange of the wavenumbers k:

$$\left.\begin{aligned} E_u(k) &= 0.75E_v(k) = 0.75E_w(k) = K_1\,\epsilon^{\frac{2}{3}}k^{-\frac{5}{3}},\\ E_t(k) &= K_2 N\epsilon^{-\frac{1}{3}}k^{-\frac{5}{3}}, \end{aligned}\right\} \tag{4.18}$$

where E_u, E_v, E_w, E_t are longitudinal (in the mean velocity direction) one-dimensional spectra of u-, v-, w- and t-fluctuations, and K_1 and K_2 are constant. The available experimental data show that apparently $K_1 \approx 0.5$ and $K_2 \approx 0.7$ but the scatter of the experimental values for these coefficients (especially for K_2) is rather large; see for example Monin & Yaglom (1975, Chap. 8), and Yaglom (1981). The time spectra of turbulent fluctuations can easily be obtained from the records of the fluctuations with the aid of the modern methods of digital spectral analysis. When the time spectra are found, the longitudinal spatial spectra can be determined with the aid of Taylor's frozen-turbulence hypothesis. The inertial range of the measured spectra can be determined then as the range of wavenumbers k where these spectra are proportional to $k^{-\frac{5}{3}}$ and, if K_1 and K_2 are known, the inertial ranges of velocity and temperature fluctuations can be used to evaluate ϵ and N.

With the aid of this method numerous values of ϵ and N were determined from the Tsimlyansk data. A sonic anemometer and fast-response thermometer were placed close to each other during the measurements and values of ϵ were calculated as the arithmetic means of the values determined from the records of all the wind components given by the anemometer; K_1 was taken to be equal to 0.5 and K_2 to 0.7

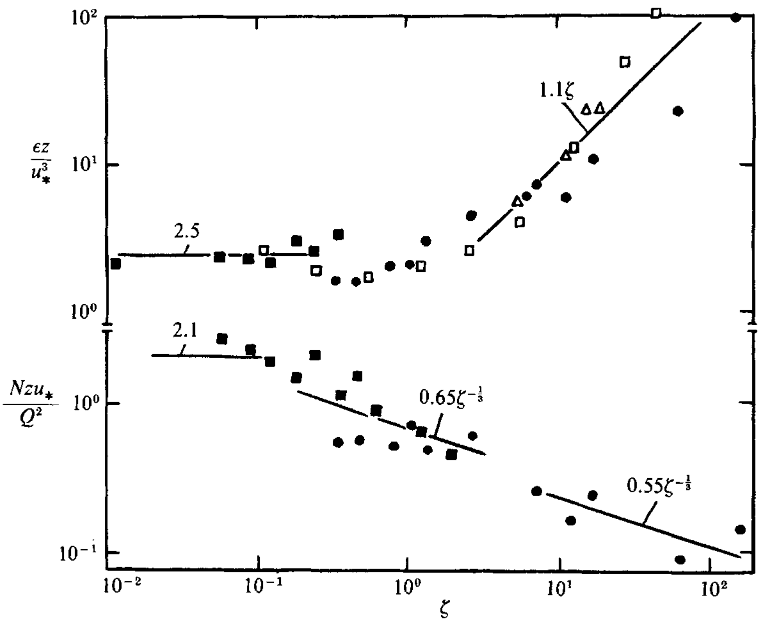

FIGURE 7. The same profiles as in figure 6 from the data of various authors. The symbols have the same meaning as in figure 4.

in these determinations. The data permit both the functions $\phi_\epsilon(\zeta)$ and $\phi_N(\zeta)$ to be plotted (figure 6). We see that the experimental data agree satisfactorily with the theoretical equations (4.12), (4.13) and (4.15) and lead to the following estimates of the coefficients C_ϵ, C_N and B_N:

$$C_\epsilon \approx 1.2, \quad C_N \approx 0.65, \quad B_N \approx 0.85. \tag{4.19}$$

Figure 7 shows the functions $\phi_\epsilon(\zeta)$ and $\phi_N(\zeta)$ determined from values of ϵ and N in an unstable ASL found in the literature. These data are rather scattered but on the whole they also do not contradict (4.12), (4.13) and (4.15) with the coefficients (4.19). We should emphasize, however, that the experimental estimates (4.19) for C_N and B_N do not agree well with (4.14) and (4.16), which by virtue of (3.5) and (4.8) imply that

$$C_N \approx 1.15, \quad B_N \approx 1.3. \tag{4.20}$$

The discrepancy between (4.19) and (4.20) characterizes the degree of accuracy of the data and equations used in the derivation of these estimates. It is clear that the experimental values of the coefficients B_7 and C_7 related to the third-order moments are rather crude but the errors in these coefficients alone cannot explain all the discrepancy. One further reason may be the inaccuracy of the budget equation (4.10) due to the horizontal inhomogeneity of the Tsimlyansk experiment site (this inhomogeneity apparently also affected the early data of Mordukhovich & Tsvang 1966). Moreover, the scatter in the experimental values of the coefficient K_2 shows that the determination of the temperature dissipation N from the inertial range of the temperature spectra can also lead to some errors. It is clear that special, more complete, measurements are needed to verify all the equations used in the present paper and to ascertain the true reasons for the discrepancies discovered.

The authors are grateful to Dr L. R. Tsvang, Dr S. L. Zubkovskii, V. G. Perepelkin, M. M. Fedorov and other experimenters at the Institute of Atmospheric Physics who obtained the data used in our paper and took part in numerous discussions in the course of this work.

REFERENCES

ARYA, S. P. S. 1972 Free convection similarity and measurements in flows with and without shear. *J. Atmos. Sci.* **29**, 877–885.

BARENBLATT, G. I. 1980 *Similarity, Self-Similarity, and Intermediate Asymptotics.* Plenum.

BATCHELOR, G. K. 1950 The application of the similarity theory of turbulence to atmospheric diffusion. *Q. J. R. Met. Soc.* **76**, 133–146.

BATCHELOR, G. K. 1952 Diffusion in a field of homogeneous turbulence. II. The relative motion of particles. *Proc. Camb. Phil. Soc.* **48**, 345–362.

BATCHELOR, G. K. 1953 *The Theory of Homogeneous Turbulence.* Cambridge University Press.

BATCHELOR, G. K. 1964 Diffusion from sources in a turbulent boundary layer. *Arch. Mech. Stosowanej* **16**, 661–670.

BERNSTEIN, A. B. 1966 A new dimensional approach to the problem of flux-gradient relationships near the ground. *Q. J. R. Met. Soc.* **92**, 560–566.

BETCHOV, R. & YAGLOM, A. M. 1971 Comments on the theory of similarity as applied to turbulence in an unstably stratified fluid. *Izv. Akad. Nauk SSSR, Ser. Fiz. Atmosf. i Okeana* **7**, 1270–1279 (829–832 in the Engl. transl. of the journal).

BINKOWSKI, F. S. 1979 A simple semi-empirical theory for turbulence in the atmospheric surface layer. *Atmos. Environ.* **13**, 247–253.

BRADLEY, E. F. & ANTONIA, R. A. 1979 Structure parameters in the atmospheric surface layer. *Q. J. R. Met. Soc.* **105**, 695–705.

BRADLEY, E. F., ANTONIA, R. A. & CHAMBERS, A. J. 1981*a* Temperature structure in the atmospheric surface layer. 1. The budget of temperature variance. *Boundary-Layer Met.* **20**, 275–292.

BRADLEY, E. F., ANTONIA, R. A. & CHAMBERS, A. J. 1981*b* Turbulence Reynolds number and the turbulent kinetic energy balance in the atmospheric surface layer. *Boundary-Layer Met.* **21**, 183–197.

BRADLEY, E. F., ANTONIA, R. A. & CHAMBERS, A. J. 1982 Streamwise heat flux budget in the atmospheric surface layer. *Boundary-Layer Met.* **23**, 3–15.

BUSINGER, J. A. 1973 Turbulent transfer in the atmospheric surface layer. In *Workshop on Micrometeorology* (ed. D. A. Haugen), pp. 67–100. American Meteorological Society.

BUSINGER, J. A., WYNGAARD, J. C., IZUMI, Y. & BRADLEY, F. F. 1971 Flux-profile relationships in the atmospheric surface layer. *J. Atmos. Sci.* **28**, 181–189.

BUSINGER, J. A. & YAGLOM, A. M. 1971 Introduction to Obukhov's paper on 'Turbulence in an atmosphere with non-uniform temperature'. *Boundary-Layer Met.* **2**, 3–6.

CALDER, K. L. 1967 A criticism of Bernstein's new dimensional approach to the flux-gradient relationships near the ground. *Q. J. R. Met. Soc.* **93**, 544–550.

CHIBA, O. 1978 Stability dependence of the vertical wind velocity skewness in the atmospheric surface layer. *J. Met. Soc. Japan* **56**, 140–142.

DONELAN, M. & MIYAKE, M. 1973 Spectra and fluxes in the boundary layer of the trade-wind zone. *J. Atmos. Sci.* **30**, 444–464.

DYER, A. J. 1967 The turbulent transport of heat and water vapour in an unstable atmosphere. *Q. J. R. Met. Soc.* **93**, 501–508.

DYER, A. J. & BRADLEY, E. F. 1982 An alternative analysis of flux-gradient relationships at the 1976 ITCE. *Boundary-Layer Met.* **22**, 3–19.

DYER, A. J., GARRATT, J. R. & FRANCEY, R. J. 1981 The international turbulence comparison experiment (Australia, 1976) – Central core data. *CSIRO, Div. Atmos. Phys., Tech. Pap.* 38.

FRANCEY, R. J. & GARRATT, J. R. 1981 Interpretation of flux-profile observations at ITCE (1976). *J. Appl. Met.* **20**, 603–618.

GARRATT, J. R., FRANCEY, R. J., MCILROY, I. C., DYER, A. J., HELMOND, I., BRADLEY, E. F. & DENMAN, O. T. 1979 The international turbulence comparison experiment (Australia, 1976) – Micrometerological support data. *CSIRO, Div. Atmos. Phys., Tech. Pap.* 37.

HAUGEN, D. A., KAIMAL, J. C. & BRADLEY, E. F. 1971 An experimental study of Reynolds stress and heat flux in the atmospheric surface layer. *Q. J. R. Met. Soc.* **97**, 168–180.

HAYASHI, M. 1974 A preliminary study on the statistical characteristics of the atmospheric turbulence. *J. Met. Soc. Japan* II **52**, 400–406.

HICKS, B. B. 1978 Some limitations of dimensional analysis and power laws. *Boundary-Layer Met.* **14**, 567–569.

HICKS, B. B. 1981 An examination of turbulence statistics in the surface boundary layer. *Boundary-Layer Met.* **21**, 389–402.

HÖGSTRÖM, U. 1985 Von Kármán's constant in atmospheric boundary layer flows: reevaluated. *J. Atmos. Sci.* **42**, 263–270.

HÖGSTRÖM, U. 1988 Non-dimensional wind and temperature profiles in the atmospheric surface layer: A re-evaluation. *Boundary-Layer Met.* **42**, 55–78.

HUNT, J. C. R. 1984 Turbulence structure in thermal convection and shear-free boundary layers. *J. Fluid Mech.* **138**, 161–184.

HUNTLEY, H. E. 1952 *Dimensional Analysis*. MacDonald (also Dover, 1966).

ISAACSON, E. & ISAACSON, M. 1975 *Dimensional Methods in Engineering and Physics*. Edward Arnold.

IZUMI, Y. 1971 Kansas 1968 field program data report. *AFCRL*-72-0041. *Environ. Res. Papers* 372. Air Force System Command.

IZUMI, Y. & CAUGHEY, J. S. 1976 Minnesota 1973 atmospheric boundary layer experiment data report. *AFCRL-TR*-76-0038. *Environ. Res. Papers* 547. Air Force Cambridge Res. Lab.

KADER, B. A. 1987 Anisotropic wind velocity and temperature fluctuations in a neutrally stratified atmospheric surface layer. In *Meteorological Researches* no. 28 (ed. L. R. Tsvang & B. A. Kader), pp. 26–35. Acad. Sci. USSR, Soviet Geophys. Committee (in Russian).

KADER, B. A. 1988 Three-layer structure of an unstably stratified atmospheric surface layer. *Izv. Akad. Nauk SSSR, Ser. Fiz. Atmosf. i Okeana* **24**, 1235–1250.

KADER, B. A. & PEREPELKIN, V. G. 1984 Wind velocity and temperature profiles in the atmospheric surface layer in neutral and unstable stratifications. *Izv. Akad. Nauk SSSR, Ser. Fiz. Atmosf. i Okeana* **20**, 151–161.

KADER, B. A. & PEREPELKIN, V. G. 1989 Effect of the unstable stratification on wind and temperature profiles in the surface layer. *Izv. Akad. Nauk SSSR, Ser. Fiz. Atmosf. i Okeana* **25**, 787–795.

KADER, B. A. & YAGLOM, A. M. 1980 Similarity laws for turbulent wall flows. In *Developments in Science and Technology, Ser. Mech. Liquid and Gas* vol. 18, pp. 3–111. Soviet Inst. of Sci. Eng. Inform., Moscow (in Russian).

KADER, B. A. & YAGLOM, A. M. 1984 Turbulent structure of an unstable atmospheric surface layer. In *Nonlinear and Turbulent Processes in Physics* (ed. R. Z. Sagdeyev), vol. 2, pp. 829–845. Harwood.

KADER, B. A. & YAGLOM, A. M. 1987 Spectra of anisotropic velocity and temperature fluctuations in turbulent wall flows. In *Problems of Turbulent Flows* (ed. V. V. Struminsky), pp. 65–74. Moscow: Nauka (Transl. *Fluid Mech.-Sov. Res.* **16** (6) 89–102, 1987).

KADER, B. A., YAGLOM, A. M. & ZUBKOVSKII, S. L. 1989 Spatial correlation functions of surface layer atmospheric turbulence at neutral stratification. *Boundary-Layer Met.* **47**, 233–249.

KAI, K. 1982 The budget of turbulent energy measured at the ERC 30-m meteorological tower. *J. Met. Soc. Japan* II **60**, 1117–1131.

KERMAN, B. R. 1978 Multi-component linear digital simulation of atmospheric boundary layer turbulence. *Ind. Aerodyn.* **3**, 39–59.

KLINE, S. J. 1986 *Similitude and Approximation Theory*. Springer.

MASSEY, B. S. 1978 Directional analysis? *Intl J. Mech. Eng. Educ.* **6**, 33–36.

MONIN, A. S. & OBUKHOV, A. M. 1954 Basic laws of turbulent mixing in the atmospheric mixing layer. *Trudy Geofiz. Inst. Akad. Nauk SSSR* no. 24 (151), 163–187.

MONIN, A. S. & YAGLOM, A. M. 1971 *Statistical Fluid Mechanics*, vol. 1. MIT Press.

MONIN, A. S. & YAGLOM, A. M. 1975 *Statistical Fluid Mechanics*, vol. 2. MIT Press.

MONJI, N. 1973 Budgets of turbulent energy and temperature variance in the transition zone from forced to free convection. *J. Met. Soc. Japan* II **15**, 133–145.

MONJI, N. 1975 Characteristics of the horizontal wind fluctuations in the surface layer under strong convective conditions. *J. Met. Soc. Japan* **53**, 99–102.

MORDUKHOVICH, M. I. & TSVANG, L. R. 1966 Direct measurements of turbulent fluxes at two heights in atmospheric surface layer. *Izv. Akad. Nauk SSSR, Ser. Fiz. Atmosf. i Okeana* **2**, 786–803.

OBUKHOV, A. M. 1946 Turbulence in thermally inhomogeneous atmosphere. *Trudy Inst. Teor. Geofiz. Akad. Nauk SSSR* no. 1, 95–115 (Transl. in *Boundary-Layer Met.* **3**, 7–29, 1971).

OBUKHOV, A. M. 1960 Structure of temperature and velocity fields under conditions of free convection. *Izv. Akad. Nauk SSSR, Ser. Geofiz.* no. 9, 1392–1396.

PANOFSKY, H. A., TENNEKES, H., LENSCHOW, D. A. & WYNGAARD, J. C. 1977 The characteristics of turbulent velocity components in the surface layer under convective conditions. *Boundary-Layer Met.* **11**, 355–361.

PANTON, R. L. 1984 *Incompressible Flow*. John Wiley.

PETUKHOV, B. S. & POLYAKOV, A. F. 1988 *Heat Transfer in Turbulent Mixed Convection*. Hemisphere.

PRANDTL, L. 1932 Meteorologische Anwendung der Strömungslehre. *Beitr. Phys. fr. Atmos.* **19**, 188–202.

PRIESTLEY, C. H. B. 1954 Convection from a large horizontal surface. *Austral. J. Phys.* **7**, 176–201.

PRIESTLEY, C. H. B. 1955 Free and forced convection in the atmosphere near the ground. *Q. J. R. Met. Soc.* **81**, 139–143.

PRIESTLEY, C. H. B. 1956 Convection from the Earth's surface. *Proc. R. Soc. Lond.* A **238**, 287–304.

PRIESTLEY, C. H. B. 1959 *Turbulent Transfer in the Lower Atmosphere*. Chicago University Press.

PRIESTLEY, C. H. B. 1960 Temperature fluctuations in the atmospheric boundary layer. *J. Fluid Mech.* **7**, 375–384.

RAYMENT, R. & CAUGHEY, S. J. 1977 An investigation of the turbulence equations in the atmospheric boundary layer. *Boundary-Layer Met.* **11**, 15–26.

SCHACHER, G. E., DAVIDSON, K. L., HOULIHAN, T. & FAIRALL, C. W. 1981 Measurements of the rate of dissipation of turbulent kinetic energy, ϵ, over the ocean. *Boundary-Layer Met.* **20**, 321–330.

TELFORD, J. W., BUSINGER, J. A. & HÖGSTRÖM, U. 1986 Comments on 'Von Kármán's constant in atmospheric boundary layer flows: reevaluated'. *J. Atmos. Sci.* **43**, 2127–2134.

TENNEKES, H. 1984 Similarity relations, scaling laws, and spectral dynamics. In *Atmospheric Turbulence and Air Pollution Modelling* (ed. F. T. M. Nieuwstadt & H. van Dop), pp. 36–68. Reidel.

VOLKOV, Y. A., KOPROV, B. M. & KRAVCHENKO, T. K. 1975 Vertical correlation functions of turbulent fields in the atmospheric surface layer. *Izv. Akad. Nauk SSSR, Ser. Fiz. Atmosf. i Okeana* **11**, 794–801.

WESELY, M. L. 1974 Magnitudes of turbulent fluctuations in the atmospheric surface layer. In *Symp. Atmos. Diff. and Air Polution, Santa Barbara, Calif.*, 1974, pp. 15–18. Am. Met. Soc.

WILLIAMS, W. 1892 On the relation of the dimensions of physical quantities in space. *Phil. Mag.* **34**, 234–271.

WYNGAARD, J. C. & COTÉ, O. R. 1971 The budgets of turbulent kinetic energy and temperature variance in the atmospheric surface layer. *J. Atmos. Sci.* **28**, 190–201.

WYNGAARD, J. C., COTÉ, O. R. & IZUMI, Y. 1971 Local free convection, similarity, and the budgets of shear stress and heat flux. *J. Atmos. Sci.* **28**, 1171–1182.

YAGLOM, A. M. 1974 Data on turbulence characteristics in the atmospheric surface layer. *Izv. Akad. Nauk SSSR, Ser. Fiz. Atmosf. i Okeana* **10**, 566–586 (341–352 in English transl. of the journal).

YAGLOM, A. M. 1979 Similarity laws for constant-pressure and pressure-gradient turbulent wall flows. *Ann. Rev. Fluid Mech.* **11**, 505–540.

YAGLOM, A. M. 1981 Laws of small-scale turbulence in atmosphere and oceans. *Izv. Akad. Nauk SSSR, Ser. Fiz. Atmosf. i Okeana* **17**, 1235–1257 (919–935 in English transl. of the journal).

YASUDA, N. 1978 High-order equations of turbulence and characteristics of turbulence based on their simple solutions in the atmospheric boundary layer near the ground. *Pap. Met. Geophys.* **29**, 109–129.

ZHANG, S. F., ONCLEY, S. P. & BUSINGER, J. A. 1988 A critical evaluation of the von Kármán constant from a new atmospheric surface layer experiment. In *Preprint Volume, Eighth Symp. on Turbulence and Diffusion*. Am. Met. Soc., Boston, Mass.

ZILITINKEVICH, S. S. 1971 On the turbulence and diffusion under free convection conditions. *Izv. Akad. Nauk SSSR, Ser. Fiz. Atmosf. i Okeana* **7**, 1263–1269.

ZILITINKEVICH, S. S. 1973 Shear convection. *Boundary-Layer Met.* **3**, 416–423.

ZUBKOVSKII, S. L. & TSVANG, L. R. 1966 On the horizontal turbulent heat flux. *Izv. Akad. Nauk SSSR, Ser. Fiz. Atmosf. i Okeana* **2**, 1307–1310.